The transition from natural to rational number knowledge

From a mathematics point of view, a rational number is commonly defined as a number that is or can be expressed in the form $\frac{a}{b}$, where $a$ and $b$ are integers, and $b$ is non-zero. Because integers, terminating decimals, and also repeating decimals can be expressed in this form, they are rational numbers. For the purposes of this chapter, we will refer to numbers that are actually in the $\frac{a}{b}$ form as fractions, and to rational numbers written in base-10 notation as decimal numbers.

A lot of research has indicated that a strong understanding of rational numbers is of critical importance for mathematics achievement in general and for performance in important content domains of the mathematics curriculum in particular, such as algebra (e.g., Siegler et al., 2012). Given that the understanding of rational numbers has been shown to relate to later mathematics achievement, it is problematic that many learners have trouble understanding several aspects of rational numbers, even until the end of secondary school and in adulthood (Vamvakoussi, Van Dooren, & Verschaffel, 2012). In this chapter we elaborate on learners’ difficulties with rational numbers. First, we discuss the importance of robust rational number understanding for learners’ general mathematics achievement. Second, we highlight the challenges that many learners face with respect to various aspects of rational numbers. Third, we elaborate on previous research that focused on a major source of learners’ difficulty, namely the natural number bias. Fourth, we introduce two complementary theoretical perspectives that have been employed in the past decade to study the natural number bias, namely the framework theory approach to conceptual change and the dual process perspective of reasoning. And then we review some studies that we conducted from these
perspectives. Finally, we discuss theoretical and educational implications of these studies and provide suggestions for further research.

The importance of rational numbers

A strong understanding of rational numbers is of critical importance for mathematics achievement in general and for particular domains of the mathematics curriculum. For example, in a study by Booth and Newton (2012), learners’ fraction knowledge was a better predictor for their algebra readiness than their whole number magnitude knowledge. Further, Siegler, Thompson, and Schneider (2011) found high correlations between upper elementary school children’s fraction magnitude knowledge and their general mathematics achievement. Siegler et al. (2012) extended these findings with longitudinal data from the U.S. and U.K. showing that 5th graders’ fraction knowledge significantly predicted their general mathematics and algebra scores in high school, even after controlling for reading achievement, IQ, working memory, whole number knowledge, family income, and family education.

Bailey, Hoard, Nugent, and Geary (2012) found that the predictive relation between learners’ rational number knowledge and their later mathematics achievement could not be explained by their general mathematical ability. As a possible explanation for this predictive relation, they mentioned that learners with a high general mathematical ability might understand rational numbers more easily than those with a low general mathematical ability. Their results showed that this alternative explanation was not the sole explanation for the predictive relation. After controlling for general mathematics achievement in 6th grade and rational number knowledge in 7th grade, followed by controlling for working memory and intelligence, they still found that 6th graders’ rational number knowledge significantly predicted mathematics achievement in grade 7.
Rational numbers: A challenge for learners and for mathematics education

Although the importance of rational number learning is widely acknowledged and considerable time and effort are invested in rational number teaching, numerous studies have documented that learners in elementary and secondary school, educated adults, and even (prospective) teachers face great difficulties with various aspects of rational numbers (see for example Cramer, Post, & delMas, 2002; Depaepe et al., 2015; Hasemann, 1981; Janssen, Verschaffel, Tuerlinckx, Van den Noortgate, & De Fraine, 2010; Lortie-Forgues, Tian, & Siegler, in press; Mazzocco & Devlin, 2008; McMullen, Laakkonen, Hannula-Sormunen, & Lehtinen, 2015; Meert, Grégoire, & Noël, 2010; Obersteiner, Van Dooren, Van Hoof, & Verschaffel, 2013; 2015 Siegler et al., 2012; Vamvakoussi, Christou, Mertens, & Van Dooren, 2011; Vamvakoussi, Van Dooren, & Verschaffel, 2012, 2013; Vamvakoussi & Vosniadou, 2004, 2010; Van Hoof, Janssen, Verschaffel, & Van Dooren, 2015; Van Hoof, Lijnen, Verschaffel, & Van Dooren, 2013; Van Hoof, Vandewalle, Verschaffel, & Van Dooren, 2015; Van Hoof, Verschaffel, & Van Dooren, 2015). To elaborate on one example, a recent study with a national representative sample of Flemish 8th graders revealed that only about one fourth succeeded in achieving the minimum standards for operations in arithmetic including rational numbers (Vlaamse Overheid, 2010). An example item used in this assessment is: “In the bookstore, customers get a 5% reduction, which makes that the book I bought is 1 euro cheaper. What was the price without the reduction?”.

There are many reasons why learners have difficulty with rational numbers, some of which are summarized by Moss (2005): a) there are several conceptually distinct meanings attached to rational numbers that need to be understood and coordinated (e.g., fractions as parts of a whole, fractions as ratios, fractions as numbers); b) new symbols and representations are introduced that, again, need to be understood and coordinated (e.g., $\frac{1}{2}$, 1.5, $\sqrt{2}$, 0.625, and 0.5).
3 \frac{1}{6}, 0.5, 0.50, 50\%); c) learners need to construct a complex knowledge network for number based on multiplicative rather than on additive relations; d) the understanding of the unit and the arithmetical operations need to be reconceptualized. All in all, there is considerable new material to be learned and the content is highly complex. Moreover, learners’ prior knowledge and experience in the domain of number is not always compatible with the new content on rational numbers, as already indicated by the above-mentioned reasons (c) and (d). As we discuss in the following paragraph, this turns out to be a major source of difficulties with rational numbers.

The interference of natural number knowledge in rational number tasks

A much noticed, studied, and reported phenomenon is the interference of natural number knowledge in learners’ attempts to make sense of rational numbers and deal with rational number tasks (see for example Kilpatrick et al. 2001; Moss, 2005; Ni & Zhou, 2005; Van Dooren, Lehtinen, & Verschaffel, 2015). It appears that learners transfer their natural number knowledge to the domain of rational numbers, using it to interpret information coming from instruction, and to solve rational number problems. The most salient indication is learners’ systematic errors that occur when their prior knowledge and experiences with natural numbers are not appropriate for the task at hand. Such tasks are commonly called incongruent. For example, if learners are asked to compare the numerical sizes of 0.32 and 0.316, they may be inclined to rely on the knowledge that 316 is larger than 32, as well as on the knowledge that 0.316 has more digits than 0.32, leading them to the wrong conclusion that 0.316 is larger than 0.32. In contrast, items where reasoning purely in terms of natural numbers leads to a correct answer are called congruent items. For instance, if learners are asked to compare the numerical sizes of 0.32 and 0.3, relying on their natural number
knowledge that 32 is larger than 3, and/or that 0.32 has more digits will lead them to the correct conclusion that 0.32 is larger than 0.3. So, incongruency emerges when there is a difference between the solutions that would emerge with the use of natural number knowledge as contrasted with rational number knowledge. Rational numbers differ from natural numbers in many and important ways, each of which may be overlooked by learners (see Stafylidou & Vosniadou, 2004, and Vamvakoussi & Vosniadou, 2010, for a detailed account). In what follows, we focus on three main differences, relating to size, operations, and density, which are reported to lead to those systematic errors.

**The size of rational numbers**

Research indicates that errors in judging or comparing the size (i.e. magnitude) of decimal numbers are frequently made because of learners’ misconception that, just as it is the case with natural numbers, the size of decimal numbers can be determined by considering differences in their number of digits (“longer decimals are larger” and “shorter decimals are smaller”) (see for example Resnick et al., 1989). This leads to systematic mistakes such as concluding that 0.32 is larger than 0.5 because 32 is larger than 5. Similarly, learners’ errors in understanding the magnitudes of fractions are found to be largely based on the numerical value of each of the components of the fractions (“a fraction’s numerical value increases when its denominator, numerator, or both increase”) (see for example Clarke & Roche, 2009; Stafylidou & Vosniadou, 2004). As with decimal numbers, this leads to systematic errors such as concluding that $\frac{5}{9}$ is larger than $\frac{3}{4}$.

**The effect of arithmetic operations**

When first learning arithmetic with natural numbers, many learners implicitly deduce that multiplication and addition will always result in a larger outcome whereas division and
subtraction will always result in a smaller outcome. In the domain of rational numbers these rules no longer necessarily apply. However, learners may still rely on them, leading to mistakes such as thinking that “0.99 * 5” leads to an outcome larger than 5 or accepting non-whole numbers as multiplicands but not as multipliers (De Corte, Verschaffel, & Van Coillie, 1988; Fischbein, Deri, Nello, & Marino, 1985; Hasemann, 1981).

**The dense structure of rational numbers**

While natural numbers are discrete (one can always point out which number comes next), rational numbers are dense; rational numbers do not have a successor number, because between any two rational numbers are always infinitely many other numbers. This difference between natural and rational numbers leads to the common assumption that there are no (or finitely many) numbers between two pseudo-successive numbers (for example between 1.2 and 1.3). Numerous studies have indicated that the dense ordering of rational numbers is difficult to grasp both for elementary and secondary school learners (Hannula, Pehkonen, Maijala, & Soro, 2006; Merenluoto & Lehtinen, 2002, 2004; Smith, Solomon, & Carey, 2005).

**Representation of numbers as an intersecting difficulty**

A fourth difficulty that intersects with these three aforementioned aspects is the representation of numbers. In this chapter we focus on two difficulties learners encounter concerning the representation of numbers and more specifically in the domain of rational numbers.

First, rational numbers can be represented by fractions and decimals, and within each of these two representational types by an infinite number of possible representations. For example, “one half” can be represented as 0.5, but also as 0.500, $\frac{1}{2}$, $\frac{8}{16}$, 50%, ... While
making meaningful connections between different representational forms is highly important in mathematics, many learners struggle with it. As stated by Ni and Zhou (2005), this is especially the case in the domain of rational numbers. Learners further tend to think of a fraction as two (natural) numbers rather than as a number in its own right (see for example Smith et al., 2005; Stafylidou & Vosniadou, 2004).

Second, several studies indicate that students have difficulty interpreting literal symbols in algebra (e.g., Kieran, 2006). When first learning algebra, many students have difficulties accepting that a literal symbol can stand for a number and is not, for example, merely an abbreviation of an object’s name (e.g., “h” for “height”) (e.g., Booth, 1984). When learners do start to make the association between literal symbols and numbers, they tend to think that a literal symbol always stands for one single number (Asquith, Stephens, Knuth, & Alibali, 2007). Moreover, research indicates that many learners have a tendency to substitute these literal symbols in algebra only with natural numbers (Christou & Vosniadou, 2005, 2012; Van Dooren & Vanvaikoussi, 2010; Van Hoof, Vandewalle, et al., 2015) and not with other types of numbers, such as rational numbers.

The natural number bias

The examples presented above are instances of natural number knowledge interference in rational number learning. Ni and Zhou (2005) coined the term “whole number bias” to refer to this phenomenon. We opt for the term “natural number bias” because unlike rational numbers, natural numbers are always positive; this specific feature may also be inappropriately assumed when dealing with rational number tasks, particularly when number are represented by literal symbols. The origin of the natural number bias — and, more specifically, whether there is an innate aspect or not — is still a matter of debate (Ni & Zhou,
There is, however, substantial consensus in the literature that natural numbers are culturally privileged (Dantzig, 1954; Greer, 2004; Ni & Zhou, 2005; Vamvakoussi et al., 2012). Indeed, natural number knowledge is supported from early on by cultural representational tools and practices such as finger counting as well as by experiences such as counting songs, rhymes, and board games. Furthermore, instruction on numbers in the first elementary school years is almost exclusively devoted to natural numbers. As a result, natural number ideas — validated and strengthened in and out of school settings — are firmly established and dominate people’s numerical reasoning, which justifies the term natural number bias.

It is important to note that the natural number bias has various manifestations, other than systematic errors in incongruent tasks (see Vamvakoussi, 2015, for a concise account). For instance, much higher accuracy levels are found in congruent rational number tasks where the correct reasoning is in line with reasoning about natural numbers (Nunes & Bryant, 2008); consider also that the part-whole interpretation of fractions — which is compatible with natural number reasoning — is typically used for the introduction of fractions because it is deemed more accessible than other aspects, such as ratio (Moss, 2005). That is, the natural number bias acts as a facilitator in contexts that are compatible (congruent) with natural number knowledge. Further, it has been noticed that biased responses are commonly generated quickly, are deemed self-evident, and may be temporarily revised if one is explicitly guided to reconsider; however, they are not necessarily eradicated and may reappear at a different moment or in a different context (Fischbein, 1987; Merenluoto & Lethinen, 2004).

Theoretical frameworks for studying the natural number bias

The interference of prior knowledge in mathematics learning in general, and in rational number learning in particular, has been studied through various theoretical lenses
(Vamvakoussi, Vosniadou, & Van Dooren, 2013). We focus on two theoretical perspectives that capture different aspects of the natural number bias. The first is the framework theory approach to conceptual change (e.g., Vosniadou, 2013; Vosniadou, Vamvakoussi, & Skopeliti, 2008) that can account for the origin and the development of the bias. The second is the dual process perspective of reasoning that can capture the processes underlying people’s responses to rational number tasks, accounting for the immediacy and the perseverance of biased responses, even when one possesses the necessary knowledge to deal with the task at hand (Evans & Over, 1996).

**The conceptual change perspective**

Thus far, most of the research on the natural number bias has interpreted this phenomenon from a conceptual change perspective, and more specifically in terms of the framework theory approach to conceptual change (Vosniadou, 1994; Vosniadou et al., 2008). The origins of the conceptual change research are situated in the field of physics and physics education. More recently, however, it has been shown that the framework theory approach to conceptual change can also be successfully applied in research concerning the learning of mathematics (see for example Vosniadou & Verschaffel, 2004; Vamvakoussi et al., 2013). Applied to the domain of number (Vamvakoussi & Vosniadou, 2010), the key assumption of this approach is that, beginning early in life, people organize their daily informal as well as formal experiences with numbers in a relatively coherent framework theory that relies on implicit assumptions about what numbers are and how they behave. As discussed above, natural numbers are culturally privileged in and out of school settings. Thus, before rational numbers are introduced in instruction, a framework theory of numbers is assumed to be in place. Within this framework theory, numbers behave essentially as natural numbers. For instance, they are discrete, they can be ordered based on their place in the count list or on
the number of digits (“longer numbers are bigger”), and they “get bigger” with addition or multiplication, and “smaller” with subtraction or division.

Learners’ initial framework theory of number acts as an interpretive filter for new information about non-natural numbers. We note that this is the defining feature of the natural number bias as defined by Ni and Zhou (2005). This presents no problems when new information is compatible with what is already known. However, incompatible information may be neglected or distorted in the process. It is assumed that learners typically employ additive mechanisms of learning to gradually incorporate new information into their initial framework theory of number. Adding incompatible information, however, destroys the coherence of the initial framework theory and results in inconsistencies and misconceptions.

A central element of the framework theory approach to conceptual change is that conceptual change is a slow and gradual process. This is because people do not revise all background assumptions of the initial framework theory at once, largely because they are typically not aware of their initial implicit assumptions (e.g., Vosniadou, 1994). In the transition from the initial framework theory of number to the scientific perspective, a special class of misconceptions appears, namely synthetic conceptions (or synthetic models). Synthetic conceptions combine elements of the initial framework theory of number with elements of the new information assimilated in the knowledge structure. An example is the synthetic conception of rational numbers as a collection of distinct sets of numbers, namely natural numbers, decimals, and fractions. One consequence is the assumption that these different types of numbers have different properties. For instance, decimals may be deemed densely ordered, whereas fractions are viewed as discrete (Vamvakoussi & Vosniadou, 2010). Another example of a synthetic conception appears in the comparison of fractions, when learners assume that the fraction with the smallest denominator is larger than a fraction with a larger
denominator (e.g., $\frac{4}{6}$ vs $\frac{1}{2}$), neglecting the relation between the terms of each fraction (e.g., Gómez, Jiménez, Bobadilla, Reyes, & Dartnell, 2014; Stafylidou & Vosniadou, 2004).

It is predicted that learners’ initial framework theories are not completely abandoned as they develop a more sophisticated understanding of number (De Wolf & Vosniadou, 2015). This is particularly relevant in the case of the initial framework theories of number, because natural numbers, although subsumed under the broader category of the rational numbers, preserve all their particular characteristics within the natural numbers set; thus initial framework theories of number remain useful as well as mathematically correct in this particular context (Vamvakoussi & Vosniadou, 2010).

The dual process perspective on reasoning

The dual process perspective originated as an attempt to explain errors committed by people on problems for which they in principle have the required knowledge to reach a correct solution (Gillard, Van Dooren, Schaeken, & Verschaffel, 2009). It postulates a distinction between intuitive or heuristic reasoning (fast, automatic, and with minimal demands of working memory) and analytical reasoning (slow, controlled, and demanding of working memory) (see for example Evans & Over, 1996). Intuitive reasoning is automatically engaged and will often lead to the correct answer, but sometimes more extensive, analytical thought processes are needed. In the latter case there are two possibilities. First, an incorrect intuitive thought process may occur without any intervention by the analytical processing system. In this case a wrong answer will be generated very quickly and without much effort. The second possibility is that the analytical processing system does intervene, evaluates the intuitively obtained answer, and attempts to inhibit it and generate an alternative answer. Depending on the success of this time-consuming and effortful analytic processing, a correct answer may
or may not be obtained. Incorrect answers, in such a dual process view, can thus be the result of either an intuitive thought process that will result in the wrong answer and is not controlled by the analytical processing system or an unsuccessful intervention of the analytical processing system (Gillard et al., 2009).

The dual process perspective is a cognitive psychological one that so far has been mainly applied to logical reasoning tasks (Stanovich & West, 2000). However, several researchers in the field of mathematics education have put forward that this perspective may be used to operationalize the notion of intuition in mathematical reasoning (see for example Babai, Levyadun, Stavy, & Tirosh, 2006; Geary, 2008; Gillard et al., 2009).

Characterizing intuition in mathematical reasoning and learning is a challenging task that was undertaken in the seminal work by Fischbein (1987). Among the defining features of intuitions according to Fischbein, there are two particularly relevant for this discussion: Intuitive judgments are fast, grounded on an immediate grasp of the situation at hand; and intuitions, once established, are robust and not easily eradicated by instruction, often coexisting with mathematically correct ideas throughout a person’s life.

Reaction times can be used to disentangle intuitive and analytical reasoning during mathematical problem solving (see for example De Neys, 2006; Gillard et al., 2009). Vamvakoussi et al. (2012) argued that when faced with rational number tasks, individuals’ first, intuitive, response is grounded in natural number reasoning. Thus, errors in rational number tasks can result from a failure to inhibit intuitive knowledge and engage in analytical reasoning. This phenomenon emerges in reaction time studies, whereby it takes longer to correctly solve incongruent problems than congruent problems, due to the needed intervention by the analytical processing system. Even when people correctly solve
incongruent problems, they still take longer than when solving congruent problems, reflecting the continued interference of natural number knowledge.

**Combining the conceptual change theory and dual process perspective to study mathematical thinking and learning**

We argue that the combination of the conceptual change theory and the dual process perspective is a promising way to study rational number understanding and the occurrence of the natural number bias. Conceptual change theory enables us to describe and explain the learner’s evolution of the mathematical concept of rational number, starting from a very naïve, intuitive natural-number based idea to the scientifically correct idea. It explains why so many learners have difficulties with several aspects of rational number understanding, specifically because they are biased by their prior knowledge of natural numbers and conceptual change is needed to fully understand the various aspects of rational numbers (see for example Vosniadou, 2013). Moreover, it allows us to describe and predict the occurrence of specific synthetic conceptions (or synthetic models). The conceptual change theory is, however, less specific about the underlying ongoing reasoning processes that take place at the moment when one is solving a particular task once a more sophisticated understanding of rational numbers is acquired. The dual process perspective allows us to do so. This approach goes further than the conceptual change theory by explaining how it is possible that learners who already have the correct, scientific idea of what a number is (and have in other words succeeded in their conceptual change of a rational number) can still make mistakes (Evans & Over, 1996). As claimed above, learners’ correct scientific idea about number may come to coexist with their initial, naïve concept. So, although learners can correctly understand number from a scientific perspective, they can still make mistakes due to the influence of
intuitive thought processes that are not or not sufficiently controlled by the analytical processing system.

Overview of our studies using both conceptual change theory and dual process perspective

In the past decade, we conducted several studies using both theoretical perspectives to increase our understanding of learners’ transition from natural to rational number understanding and the role of the natural number bias in this transition. While paper-and-pencil tests and interviews are the two most common methods to investigate the natural number bias, we also applied a third, less common method, namely collecting reaction time data. As stated above, learners can correctly understand a concept, but still make mistakes due to the intrusion of incorrect, intuitive knowledge. We investigated whether this is also applicable in the case of rational number understanding, by collecting reaction time data in several of our studies.

Size

In a cross-sectional study, Van Hoof et al. (2013) investigated Flemish 7th and 11th graders’ natural number bias by measuring their accuracy levels and reaction times on fraction comparison tasks in which the fractions either had a common denominator (congruent items, for example $\frac{11}{26}$ vs. $\frac{17}{26}$) or a common numerator (incongruent items, for example $\frac{10}{27}$ vs. $\frac{10}{21}$). Accuracy was high for all problems, thus preventing the evaluation of a potential natural number bias, but this did emerge in the problem solving reaction times. Incongruent problems took longer to solve correctly than congruent problems for students in both grades. Further, the natural number bias did not decrease (neither in terms of accuracy nor in terms of reaction time) between 7th and 11th grade. Similar results have been obtained with educated adult participants. Specifically, Vamvakoussi et al. (2013) tested a group of Flemish university
students in fraction comparison tasks. Both fractions with and without common components were included in the study (see Table 1 for examples). In the comparison tasks including fractions with common components, no effect of congruency was found in students’ accuracy, due to ceiling levels. However, a significant effect of congruency was again found for reaction times: Similar to the secondary school students, the university students needed more time to respond correctly to incongruent items (i.e., fractions with common numerators). Concerning the fractions without common components, congruency had no significant effect either on accuracy or on reaction time.

**Insert Table 1**

Obersteiner et al. (2013) went further by assessing mathematics and computer science faculty members on the same type of tasks. They were all faculty members of a Flemish university. The results indicated that, similar to secondary school students and college students, congruency had a significant effect on reaction time when the fractions had common components (denominators or numerators). More specifically, experts needed more time to accurately solve incongruent tasks (e.g., $\frac{4}{17}$ vs. $\frac{4}{39}$) than congruent tasks (e.g., $\frac{22}{49}$ vs. $\frac{18}{49}$). In fraction comparison tasks without common components, congruency had no significant effect on accuracy or on reaction time data.

**Operations**

Based on the work of Christou and Vosniadou (2005), Van Hoof, Vandewalle, et al. (2015) asked Flemish 8th graders to evaluate the accuracy of algebraic expressions, some of which were congruent (i.e., interpreting the letter as a natural number leads to a correct answer; for example judging that “$x < x \times 4$” can be true) and others incongruent (i.e., interpreting the letter as a natural number leads to an incorrect answer, for example judging
that “x * 5 < x” cannot be true). Higher accuracy rates for congruent items than for incongruent items were found. Additional interviews, in which the students were asked after each item to explain how they solved the item, showed that these learners used similar strategies when they incorrectly solved incongruent items as when they correctly solved congruent items. In both cases, they almost always either referred to a natural number rule (for example: division always makes smaller) or substituted the unknown term with one or more natural numbers. While the natural number bias was manifest in multiplication and division items, only a non-significant trend was found for addition and subtraction items. The same researchers administered the same algebraic expressions to a large group of Flemish 10th and 12th graders. Again an overall natural number bias was evidenced by the higher accuracy levels on congruent than on incongruent items. As with the 8th graders, traces of the natural number bias were found only in multiplication and division items. Based on the odds ratios for the difference in accuracy between congruent and incongruent items, results indicated that the strength of the natural number bias did not decrease from grade 8 to grade 12.

In a related study, Vamvakoussi et al. (2013) asked university students to judge the correctness of algebraic statements that could be considered as either congruent (for example: “5 + 2x can be bigger than 5”) or incongruent items (for example: “5 ÷ x can be bigger than 5”). The natural number bias was reflected in the significantly lower accuracy levels on incongruent than on congruent items; and also in the significantly longer reaction times needed to answer correctly to incongruent than to the congruent items. Unlike the secondary school students studied by Van Hoof, Vandewalle, et al. (2015), the college students who participated in this study showed similar results for addition and subtraction as those found for multiplication and division. Obersteiner et al. (in press) compared the performance of German secondary school students and Flemish expert mathematicians when they were asked
to judge the correctness of algebraic expressions that were similar to the ones described in the studies above (Vamvakoussi et al., 2013; Van Hoof, Vandewalle, et al., 2015). Unlike the studies described above, the items were presented in two blocks that consisted of the same set of items, but in a first block, $x$ was defined as a natural number, while in the second block, $x$ was defined as a rational number. Results indicated that for secondary students accuracy was higher in the natural number block compared to the rational number block. Moreover, within the rational number block, they had higher accuracy levels on congruent items compared to incongruent items. The experts, in contrast, did not show traces of the natural number bias. Within the rational number block, no differences could be found for either accuracy or reaction times across congruent and incongruent items. Moreover, the experts were more accurate in the rational number block compared to the natural number block. This finding appears to be in contrast with the findings of Obersteiner et al. (2013) that showed traces of the natural number bias in a population of experts. However, the difference of the tasks has to be considered: It appears that in the context of algebraic expressions, the experts did not draw on their number knowledge; rather, they relied on their knowledge about the solvability of inequalities. As Obersteiner et al. (in press) concluded, it is possible for experts to circumvent the problem of the natural number bias when it is possible to employ strategies that do not rely on knowledge about numbers.

**Density**

Vamvakoussi and Vosniadou (2004, 2007, 2010) investigated secondary students’ (grades 7th and 9th, and 11th) understanding of the dense ordering of rational numbers. Vamvakoussi and Vosniadou (2010), administered to Greek 7th graders, 9th graders, and 11th graders a multiple-choice questionnaire consisting of items asking “How many numbers are there between $x$ and $y$?”, varying systematically the type of the $x$ and $y$ end points over natural
numbers (“How many numbers are there between 0 and 1?”), decimals (“How many numbers are there between 0.1 and 0.2?”), and fractions (“How many numbers are there between $\frac{1}{3}$ and $\frac{2}{3}$”). As predicted by the framework theory approach, they found that the idea of discreteness, a fundamental background assumption of learners’ initial framework theory of number, was very robust, leading to considerable percentages of responses of the type “there is a finite number of intermediates between two given numbers”, even for the older participants. Furthermore, the type of interval endpoints had a large effect on learners’ judgments regarding the number as well as the type of numbers in a given interval. Specifically, learners were more inclined to answer that there is an infinite number of intermediates between two natural numbers, but less so in the case of decimals and fractions. Moreover, learners were reluctant to accept that there can be decimals between two fractions or that there can be fractions between two decimals. In addition, some learners changed their answers depending on the intervals end points. For instance, infinitely many intermediates were accepted between decimals, but a finite number of intermediates between fractions. Vamvakoussi and Vosniadou (2010) attributed these findings to the synthetic conception discussed above of the rational numbers as a collection of distinct, unrelated sets of numbers.

Vamvakoussi et al. (2011) replicated the previous study with Flemish 9th graders, and compared the results with those of the 9th grade Greek students. Although the Flemish participants outperformed their Greek peers, their patterns of responses were very similar. This latter result provided further evidence for the framework theory approach to conceptual change. Indeed, this study indicated that learners coming from different educational systems face the same conceptual difficulties regarding the dense structure of rational numbers.
Finally, the density items used in the study by Vamvakoussi et al. (2012) proved the most challenging for the college students who participated in the study, compared to the comparison and operation tasks. As predicted by the dual process perspective, congruency for density items had a significant effect on accuracy (incongruent items elicited more errors), and participants took longer to respond correctly to incongruent items.

**How are the three aspects related to each other?**

Based on the existing literature and an analysis of the Flemish mathematics curriculum, Van Hoof, Verschaffel, et al. (2015) constructed and validated a comprehensive test instrument that enabled them to compare the strength (i.e., the ratio of correctly solved congruent to correctly solved incongruent solved items) of the natural number bias for the three different aspects of this bias (size, operations, and density). By administering this test to a large group of Flemish 4th to 12th graders, they found a clear manifestation of the natural number bias in terms of higher accuracy rates for congruent than for incongruent items. Second, this manifestation of the bias was equally strong in tasks with decimal numbers as with fractions. This is an interesting finding, because the available theoretical and empirical literature contains evidence that different kinds of natural number-based errors may occur in items involving these two representations (for more information, see Van Hoof, Verschaffel, et al., 2015). Third, learners’ natural number-based errors decreased across grades and the natural number bias declined over a period of at least four years (from 6th grade to 10th grade), however without completely disappearing at the end of secondary education (see Figure 1). Fourth, the strength of the natural number bias was weakest in size tasks, somewhat stronger in operations tasks, and by far the strongest in density tasks (see Figure 1). Fifth, while there was an improvement in learners’ understanding of all three aspects, they continued to struggle particularly with the operations and density aspects.
Moreover, in a study including a large group of Flemish 4th graders Van Hoof, Janssen, et al. (2015) conducted item response theory (IRT) analyses to determine if 4th graders’ ability to inhibit the inappropriate use of natural number knowledge in rational number tasks can be considered a single construct. Learners were given the Rational Number Sense Test (Van Hoof, Verschaffel, et al., 2015) that includes both congruent and incongruent items within the three aspects of the natural number bias. The finding that a Rasch scale fitted the data well is consistent with the ability to inhibit the natural number bias defining a single construct. If, on the contrary, results would have shown that 4th graders’ ability was a multidimensional construct, this would have meant that learners develop an understanding of one aspect relatively independently of developing understanding of another aspect. However, this was not the case. A closer look at the distribution of the items on the IRT-scale revealed that most density items were located at the top of the scale, meaning that they had the highest difficulty level, whereas most size items were located at the bottom of the scale, implying that they had the lowest difficulty levels, and the operations items were distributed all over the scale. These findings indicated that a good understanding of the size of rational numbers forms a prerequisite for gaining understanding in the aspect of operations and that understanding of the aspects of size and operations is a prerequisite for gaining understanding in the dense structure of rational numbers. A similar finding was obtained by Vamvakoussi et al. (2012) with college students.

**Conclusion and Future Directions**

**Summary**
In this chapter we presented a set of studies investigating the phenomenon of natural number knowledge interference in rational number learning and reasoning, that is, the natural number bias. We used two theoretical perspectives: the framework theory approach to conceptual change (Vamvakoussi et al., 2011; Vamvakoussi et al., 2012, 2013; Vamvakoussi & Vosniadou, 2010; Van Hoof, Janssen, et al., 2015; Van Hoof, Vandewalle, et al., 2015; Van Hoof, Verschaffel, et al., 2015), and the dual process perspective on reasoning (Obersteiner et al., 2013, in press; Vamvakoussi et al., 2013; Van Hoof et al., 2013). The former predicts that the transition from the initial framework theory of numbers as natural numbers to the scientific perspective of numbers as rational number is a slow process, bugged by systematic errors. Initially, learners make mistakes that reflect the direct application of natural number knowledge to rational number tasks (e.g., longer decimals are bigger; the bigger the terms, the bigger the fraction; there is no other number between $\frac{2}{5}$ and $\frac{3}{5}$; etc.). As instruction on rational numbers continues, inconsistencies and synthetic conceptions appear.

From the second perspective, the dual process model of reasoning, the natural number bias is also evident when learners make correct inferences, through an increase in the time needed to solve rational number problems that are inconsistent with natural number properties. It is also predicted that learners might be inconsistent in the sense that the same kind of task sometimes elicit a correct response and other times an incorrect one. The latter case indicates a failure to inhibit the natural number bias.

The study by Van Hoof, Verschaffel, et al. (2015) indicated that learners’ rational number understanding develops over a period of at least four years, but not continuously. The development is characterized by periods of rapid development (for example between 6th and 8th grade) and periods where it tends to stagnate (for example between 10th and 12th grade). This finding cannot be entirely explained by the relative focus on rational numbers in the
curriculum. If this was the case, then one would expect a clear evolution between 4th and 6th grade, which was not found. Mistakes that can be directly attributed to the natural number bias tended to disappear for size (for example thinking that 0.12 is larger than 0.5, because 12 is larger than 5) at the beginning of secondary education. This result does not necessarily mean that learners are no longer affected by the bias. In fraction comparison tasks, secondary school students take more time to respond correctly to incongruent $\frac{10}{27}$ vs. $\frac{10}{21}$ than congruent $\frac{6}{27}$ vs. $\frac{10}{27}$ tasks (Van Hoof et al., 2013), and the same holds for college students (Vamvakoussi, et al., 2012) and even experts (Obersteiner et al., 2013).

Furthermore, the fact that natural number based errors practically disappear overall, does not mean that learners do not make mistakes. New kinds of mistakes become prevalent. For example, Vamvakoussi et al. (2012) reported that the college students participating in their study chose the “shorter” of two decimals as bigger (e.g. 2.6 vs. 2.899). It appears that the participants retained the idea that the magnitude of a decimal depends on the number of the digits (as with natural numbers), but also used information coming from instruction [possibly from instruction on fractions, as argued also by Peled and Awawdy-Shahbari (2009) and Resnick et al. (1989)]. In this sense, this is a synthetic conception from the perspective of the framework theory approach to conceptual change. Another example can be found in the study of Gómez and Dartnell (2015). They report a meta-analysis of five datasets of fraction comparison studies, and found, especially in the older age groups, a reverse bias in fraction comparison tasks for fractions with different numerators and denominators. Congruent items (e.g., $\frac{5}{7}$ vs. $\frac{1}{3}$) elicited more errors than incongruent items (e.g., $\frac{11}{25}$ vs. $\frac{8}{13}$). Moreover, the participants needed more time to correctly solve the congruent items compared to the incongruent items. Gómez and Dartnell propose that the participants retained the idea that a
smaller denominator means a larger fraction.

Van Hoof, Verschaffel, et al. (2015) also showed that natural number bias based mistakes decrease throughout primary and secondary education for the aspect of arithmetical operations (e.g., 72 * 0.99 is larger than 72). However, such mistakes do not decrease in the last years of secondary school and remain present in college students, who also take longer to respond to incongruent tasks than to congruent tasks (Vamvakoussi et al., 2012, 2013). It should be also taken into consideration that the overall increase in accuracy does not necessarily mean that individual learners respond correctly in a systematic way (e.g., Vamvakoussi et al., 2012). Interestingly, the college students in the latter study appeared biased also with respect to addition and subtraction, whereas the secondary school students in the Van Hoof, Vandewalle, et al. (2015) study appeared biased only with respect to multiplication and division. Professional mathematicians on the other hand did not appear to be biased either in terms of accuracy, or in terms of reaction times (Obersteiner et al., in press) for any of the four arithmetic operations, presumably because they do not draw on natural number-related knowledge but rather rely on knowledge of the solvability of inequalities.

By far the most challenging aspect is the density property of rational numbers (Vamvakoussi et al., 2012; Van Hoof, Verschaffel, et al., 2015). A detailed account of the intermediate states of understanding this property was initially provided by Vamvakoussi and Vosniadou (2010), and further confirmed by the cross-cultural comparative study of Vamvakoussi et al. (2011). It appears that learners coming from different educational systems, regardless of an overall difference in performance, still face the same conceptual difficulties regarding the dense ordering of rational numbers.

The study of Van Hoof, Janssen, et al. (2015) indicated that, at least in 4th graders,
understanding the size of rational numbers forms a prerequisite for gaining understanding in the aspect of operations, for example how arithmetic operations can differentially affect the outcomes for whole numbers (e.g. multiplication produces an answer that is greater than the multiplier and multiplicand) and rational numbers (e.g., multiplication can result in a smaller answer) and that both understanding of the aspect of size and operations is a prerequisite for gaining understanding in the dense structure of rational numbers. The same finding has been obtained by Vamvakoussi et al. (2012) with college students. This finding further is consistent with and expands the integrated theory of numerical development (Siegler et al., 2011), which states that understanding magnitudes forms a crucial step in the understanding of fractions. Taken together, these studies show that rational number learning is a slow and difficult process in which the natural number bias interferes. This bias shows up in terms of errors, of synthetic conceptions, but also in terms of reaction times, even when learners respond correctly, which is an indication of the intuitive character of the natural number bias. It is also persistent, even for experts, for certain tasks.

These findings confirmed the claim made by researchers nowadays that when conceptual change has occurred and learners have acquired a scientifically correct idea, this does not mean that the initial naïve idea no longer exists. On the contrary, the scientifically correct idea may coexist with the initial concept (DeWolf & Vosniadou, 2015). Therefore, when solving incongruent rational number tasks, learners need to first inhibit their natural number based idea of a rational number to come to the right answer, which lead to longer reaction times to give a correct answer to incongruent rational number tasks.

**Future Directions**

Several of the studies reported in this chapter attempted to capture the development of the natural number bias. However, these were mainly cross-sectional studies. The
instrument developed by Van Hoof, Verschaffel, et al. (2015) could be used in longitudinal studies to capture learners’ rational number development over time and to investigate whether general developmental patterns can be found. As in the study of McMullen et al. (2015), data can be analyzed using latent profile analysis and latent transition analysis, because these analysis techniques allow us to map both gradual as well as abrupt changes in learners’ conceptual structures. While these statistical techniques are novel in the research field of conceptual change theory, they have already proven useful in a few studies (see for example McMullen et al., 2015; Schneider & Hardy, 2013). Moreover, a more detailed analysis of learners’ mistakes at the intermediate states of understanding size, operations, and density is necessary to provide an account of synthetic misconceptions predicted by the framework theory approach to conceptual change.

In the aforementioned studies, the intuitive character of learners’ reasoning processes when they deal with rational number tasks was investigated. We note that the dual process perspective offers more methodological tools than have taken advantage of thus far (Gillard et al., 2009). Experimental manipulations that could be used to reveal the natural number bias and dual process mechanisms include restricting solution time and increasing working memory load, which are assumed to hinder analytic reasoning.

Another question that cannot be answered by reaction time studies is whether the learners experience an explicit conflict between their intuitions and the valid rational number knowledge when they make errors. This is an interesting question because—depending on the answer—errors due to the natural number bias can be attributed to a failure to detect the conflict between the intuitive, erroneous response and the mathematically correct answer, or to a failure to discard the initial, tempting intuition (De Neys, Moyens, & Vansteenkoven, 2010). Data about people’s self-assessment of certainty, think-aloud protocols, but also
methodologies coming from neuroscientific research, such as monitoring participants’ skin conductance responses while they solve congruent and incongruent tasks could be valuable for establishing conflict detection.

In addition, more cross-cultural comparative studies are necessary in order to investigate the extent to which the development of learners’ concept of rational number has certain universal features, or whether the development is qualitatively different when another kind of rational number instruction is given.

Moreover, as stated by Gómez et al. (2014), we still do not know enough about the extent to which instructional interventions can moderate the effect of the natural number bias, or even help learners overcome it (at least in terms of errors). There are, of course, studies aiming at remedying learners’ natural number misconceptions with respect to one of the aspects discussed above (see Vamvakoussi et al., 2013); and there are also intervention studies that take into consideration the effects of prior natural number knowledge (e.g., Cramer et al., 2002; Moss, 2005). However, systematic, longitudinal interventions aiming at moderating the effects of the natural number bias are lacking.

There are, however, several principles for instruction stemming from the conceptual change perspective on learning that remain to be validated by further research (see, for example, Greer, 2004; Vosniadou, Ioannides, Dimitrakopoulou, & Papademetriou, 2001; Vosniadou & Vamvakoussi, 2006). One of these principles is particularly relevant to this chapter, because it pertains to the distinction between intuitive and analytic reasoning: Learners should be encouraged to become aware of their background assumptions regarding numbers; and the differences between natural number and rational numbers should be made explicit to learners, for example through refutational texts. As stated by Tippett (2010), a refutational text consists of two parts: The first part is a statement of a commonly held
misconception and the second part consists of an explicit refutation of that misconception while emphasizing the correct scientific explanation. There is a substantial body of literature indicating the effectiveness of refutational text for addressing learners’ misconceptions (for an overview, see Tippett, 2010). These steps are necessary for learners to be able to “stop and think” when dealing with rational number tasks, that is, to critically evaluate their first, intuitive response, and engage in analytic reasoning.

However, current instructional practices, at least in Flanders and in Greece, do not include these techniques. For example, Debou and Verschetze (2012) found that the most commonly used textbooks in the Flemish classroom devote very little time to the (conceptual) differences between natural and rational numbers. On the contrary, they tend to capitalize on the continuity with natural numbers in their instruction about rational numbers by emphasizing the similarities between both types of numbers. For example, a common way to teach multiplication is through the model of repeated addition, and the common way to teach division is through the model of equal sharing. Because these two models of operations cannot always be applied in tasks with rational numbers, a need for a stronger awareness of the possible negative consequences of introducing multiplication and division only using these two models is needed (Greer, 1994). Trying to build rational number knowledge on natural number ideas without emphasizing the differences between both types of numbers too may also have adverse results, because it validates and thus reinforces learners’ reliance on natural number reasoning (Moss, 2005; Vamvakoussi, 2015). Careful consideration is required in order to decide when to emphasize the dissimilarities between the natural number system and the rational number system and when to point out the appropriate, deep similarities between both number systems to help learners to gain a stronger understanding of rational numbers.
One such similarity is that natural and rational numbers alike measure quantities, they have magnitudes, and they can be placed on number lines (Schneider & Siegler, 2010; Siegler et al., 2011; Vamvakoussi & Vosniadou, 2010). We found that a good understanding of size is a prerequisite for gaining understanding of operations and possibly of density. It appears that understanding of rational number magnitude is an essential step in the understanding of rational numbers in general. This claim is further corroborated by recent studies (see for example Gabriel et al., 2012, 2013).

References


Obersteiner, A., Van Dooren, W., Van Hoof, J., & Verschaffel, L. (2013). The natural number bias and magnitude representation in fraction comparison by expert mathematicians. *Learning and Instruction, 28*, 64-72. doi:10.1016/j.learninstruc.2013.05.003


Table 1
Examples of congruent and incongruent fraction comparison tasks, both with and without common components

<table>
<thead>
<tr>
<th>Congruent</th>
<th>Incongruent</th>
</tr>
</thead>
<tbody>
<tr>
<td>Common components</td>
<td></td>
</tr>
<tr>
<td>(\frac{2}{5}) vs. (\frac{3}{5})</td>
<td>(\frac{1}{8}) vs. (\frac{1}{3})</td>
</tr>
<tr>
<td>No common components</td>
<td>(\frac{1}{4}) vs. (\frac{19}{20})</td>
</tr>
</tbody>
</table>
Figure 1: Overall change and change per aspect of the strength of the natural number bias as represented by the odds ratio (and 95% confidence interval) of accuracy for congruent and incongruent items (an odds ratio of 1 indicates that the observed accuracies for both types of items yield no evidence for the natural number bias)